Surprises in the O(N) models: Nonperturbative fixed points, large N limit and multi-criticality

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O(N) models

- They have played an important role in our understanding of second order phase transitions.

- N-component vector order parameter
  N=1…Ising, N=2…XY, N=3…Heisenberg Model

- The playground of almost all the theoretical approaches… Exact solution (2d Ising), Renormalization group（d=4-ε, 2+ε expansion), conformal bootstrap

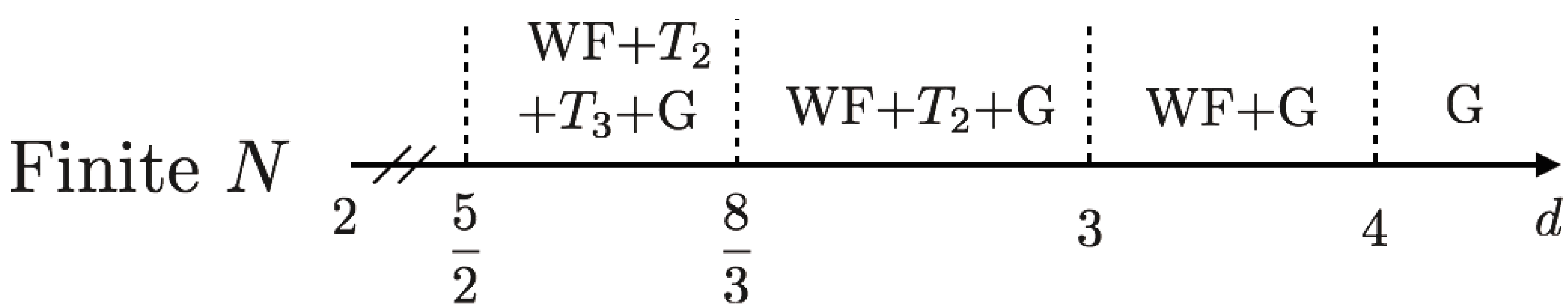
Everything is known about the criticality of O(N) models?
…This is what we want to challenge in this work.

Common wisdom on the criticality of O(N) models (finite N case)

GLW Hamiltonian $$H[\phi] = \int_x (\nabla \phi_i)^2 + U(\phi_i)$$

$$U(\phi_i) = a_2(\phi_i)^2 + a_4(\phi_i)^2 + a_6(\phi_i)^6 + \cdots$$

Below the critical dimension $d_n = 2 + 2/n$, the ϕ^{2n} term becomes relevant around the Gaussian FP (G).



A nontrivial fixed point T_n with n relevant (unstable) directions branches from G at d_n. (Wilson-Fisher FP, which describes second order phase transition, at d=4 and the tricritical FP T_2 at d=3….)

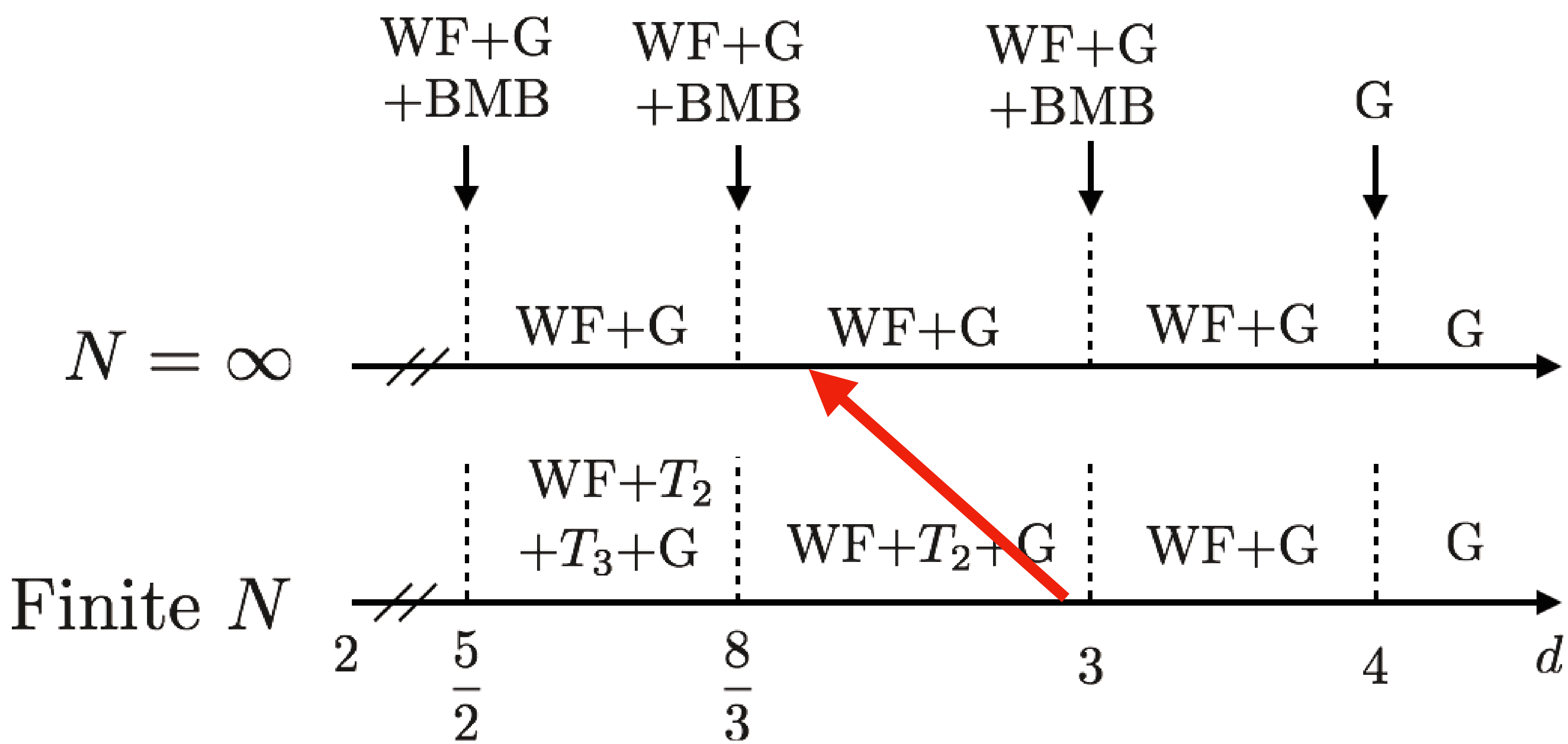
Common wisdom on the criticality of O(N) models at $N = \infty$

- At $N = \infty$, in generic dimensions 2<d<4, only Gaussian (G) and Wilson-Fisher (WF) FPs have been found.

- Exceptional case: At $d_n = 2 + 2/n$, there exists a line of FPs starting from G and it terminates at BMB (Bardeen-Moshe-Bander) FP.

- LPA of NPRG is believed to be exactly soluble.

Summary of common wisdom and a simple paradox



- What occurs if we follow T_2 from $(d = 3^-, N = 1)$ to $(d = 2.8, N = \infty)$ continuously as a function of (d,N)?…It seems that nobody asked this question!

Possible scenarios

- T_2 disappears. (Collision with another FP?)
- T_2 becomes singular.

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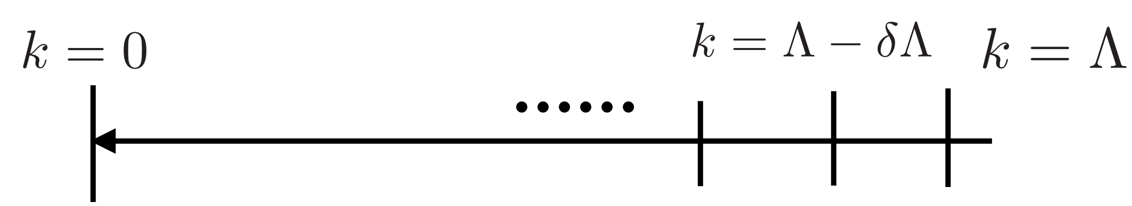
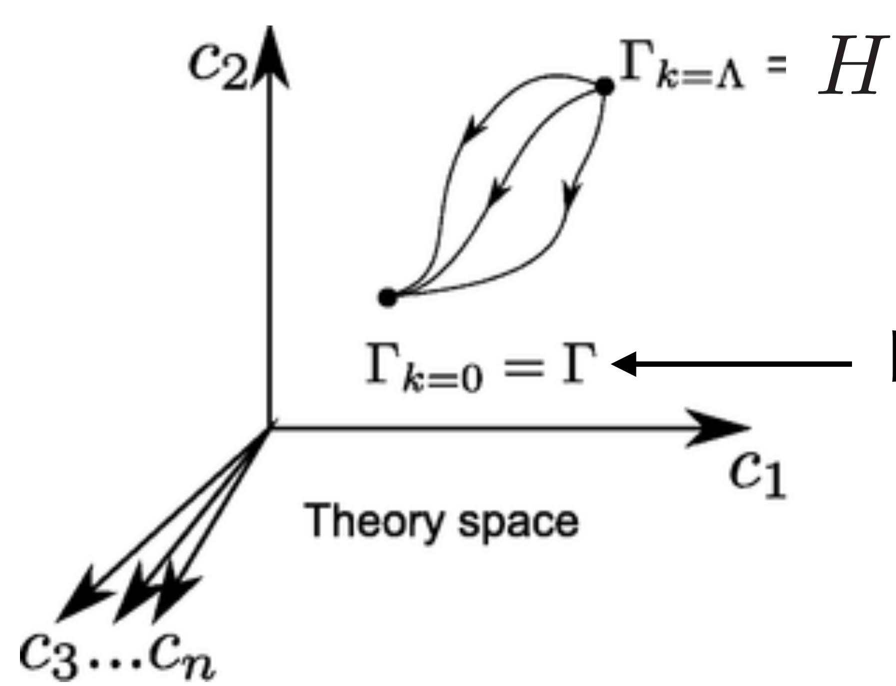
We shall see that both possibilities are realized depending on the path followed from $(d = 3^-, N = 1)$ to $(d = 2.7, N = \infty)$.

Non perturbative renormalization group (NPRG)

- Modern implementation of Wilson's RG that takes the fluctuation into account step by step in lowering the cut-off wavenumber k, in terms of wavenumber-dependent effective action Γ_k





Microscopic Hamiltonian

Effective action (Free energy), where all the fluctuations are taken into account.

NPRG equation

NPRG equation (Wetterich, Phys. Lett. B, 1993) is

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}[\partial_t R_k(q^2)(\Gamma_k^{(2)}[q,-q;\phi] + R_k(q))^{-1}]$$

$$t = \ln(k/\Lambda)$$

Derivative expansion(DE2)

- It is impossible to solve the NPRG equation exactly and we have recourse to approximations,

$$\Gamma_k[\boldsymbol{\phi}] = \int_x \left(\frac{1}{2} Z_k(\rho)(\nabla\phi_i)^2 + \frac{1}{4} Y_k(\rho)(\phi_i\nabla\phi_i)^2 \right.$$
$$\left. + U_k(\rho) + O(\nabla^4) \right) .$$

$$\rho = \phi_i\phi_i/2$$

- Simpler approximations…LPA(η =0), LPA' approximation

$$Y_k(\rho) = 0$$

$$Z_k(\rho) = \bar{Z}_k$$

$$\downarrow$$

$$\eta_t = -\partial_t \log \bar{Z}_k$$

Scaled NPRG equation

- Fixed point is found by nondimensionalized renormalized field

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

Litim cutoff $\quad R_k(q^2) = Z_k k^2 y r(y) \qquad r(y) = (1/y - 1)\theta(1-y)$

$$y = \frac{q^2}{k^2}$$

Under LPA approximation,

Finite N equation

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\, \tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\, \tilde{U}_t'(\tilde{\phi}) + (N-1)\, \frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

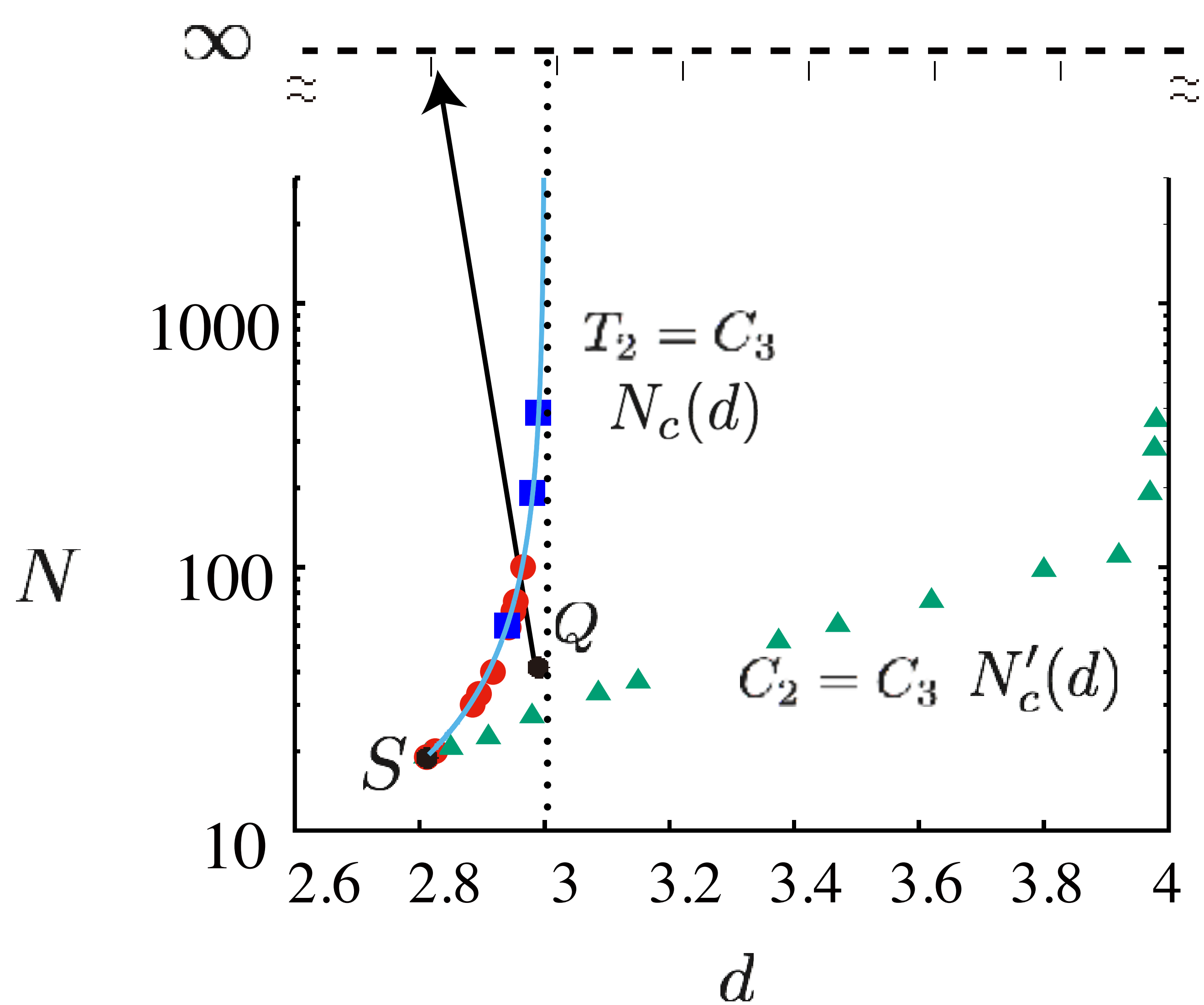
Rescaled finite N equation

$$\tilde{\phi} = \sqrt{N}\bar{\phi} \qquad \tilde{U}_t = N \bar{U}_t$$

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\, \bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\, \bar{U}_t'(\bar{\phi}) + \left(1 - \frac{1}{N}\right) \frac{\bar{\phi}}{\bar{\phi} + \bar{U}_t'(\bar{\phi})} + \frac{1}{N} \frac{1}{1 + \bar{U}_t''(\bar{\phi})}$$

Fixed point structure

We found two nonperturbative fixed points C_2 (two-unstable) and C_3 (three-unstable), which do not coincide with G at any d.



$$N = N_c(d)$$

T_2 and C_3 collide and vanish

$$N = N'_c(d)$$

C_2 and C_3 collide and vanish

The two lines meet at S=(d=2.8,N=19)

The line $N = N_c(d)$

- We can fit this line as $N_C(d)=3.6/(3-d)$.

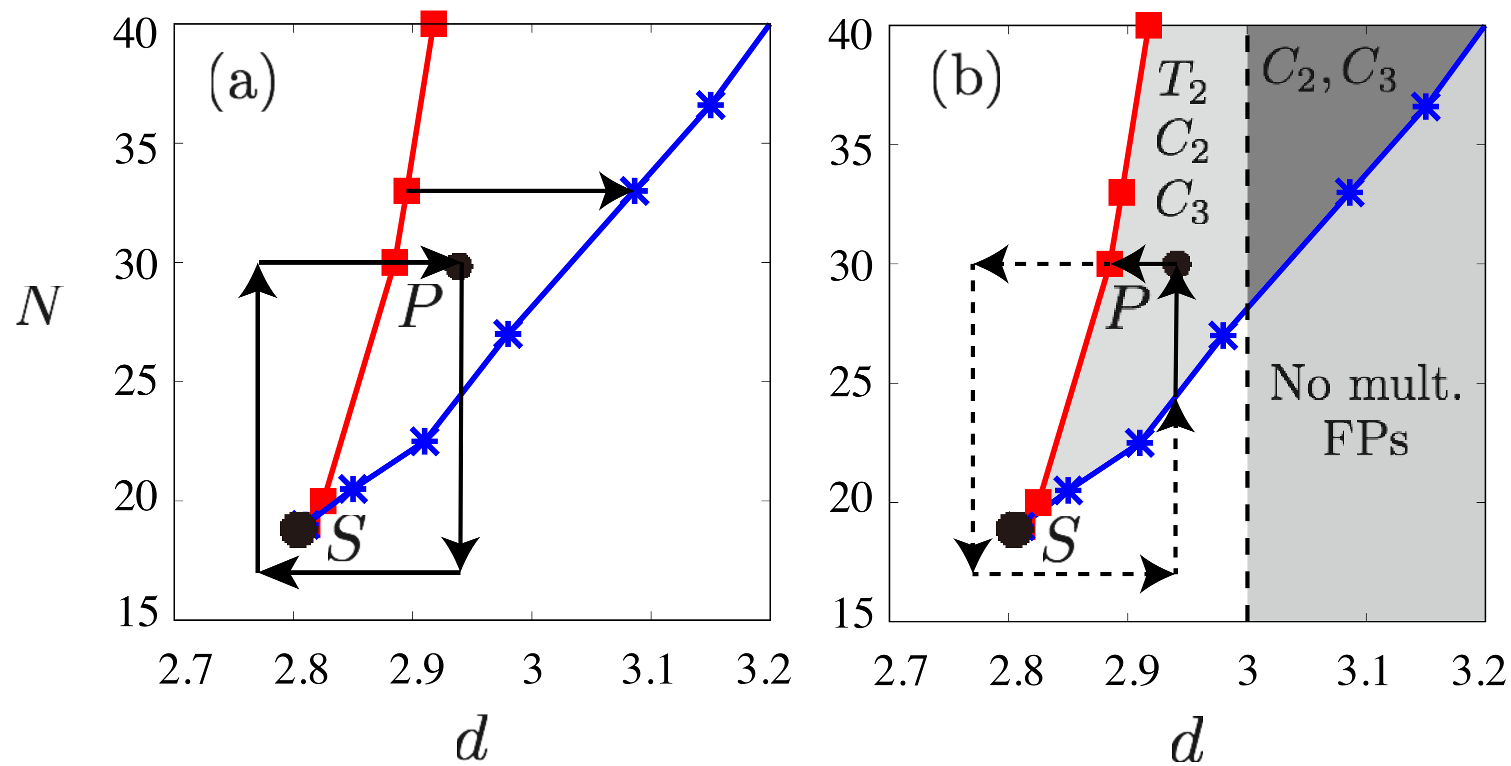
- Pisarski (1982 PRL) and Osborn-Stergiou (2018 JHEP) studied ϕ^6 theory perturbatively and showed that T_2 can exist for

$$N \leq N_c^{PT}(d) = \frac{36}{\pi^2(3-d)} \simeq \frac{3.65}{(3-d)}$$

which agrees with our numerical fit within numerical uncertainty.

- The perturbative calculation does not capture the nonperturbative FP C_3 .

Double-valued structure

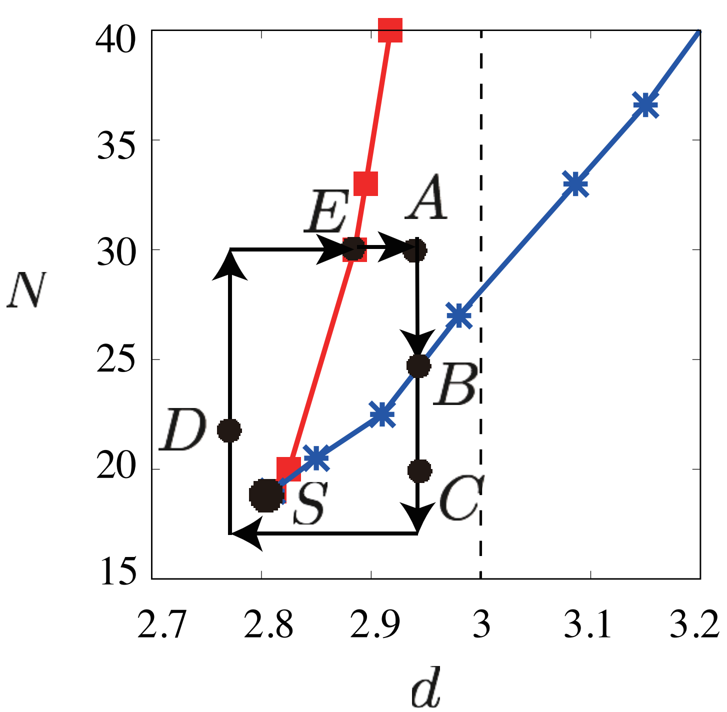


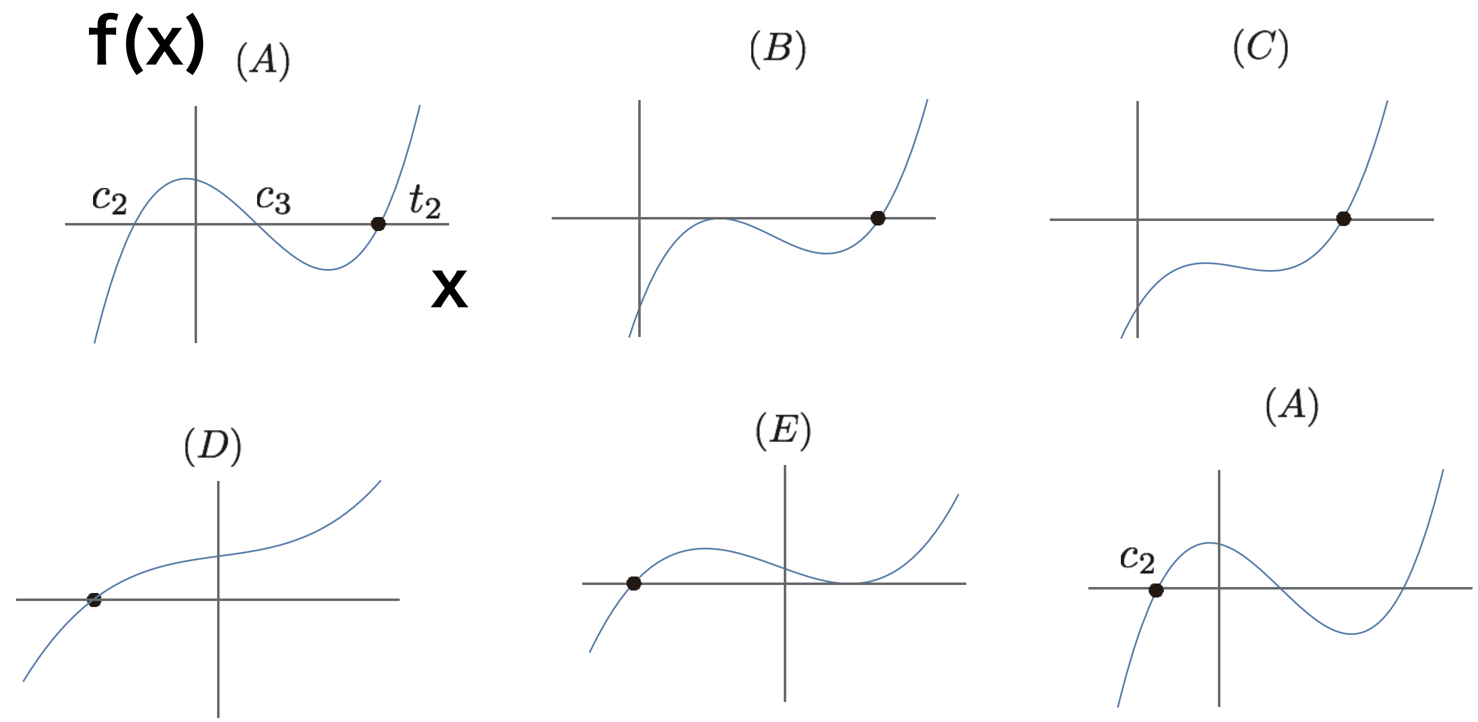
- Starting from P, we follow T_2 around a path around the point S clockwise. After full rotation it becomes C_2.

- Anticlockwise path… T_2 vanishes at $N = N_c(d)$ and it remains complex all along the dashed path. It becomes real at $N = N_c'(d)$ and comes back as C_2.

After two full rotations we go back to the same FP

Toy model for the double valued structure



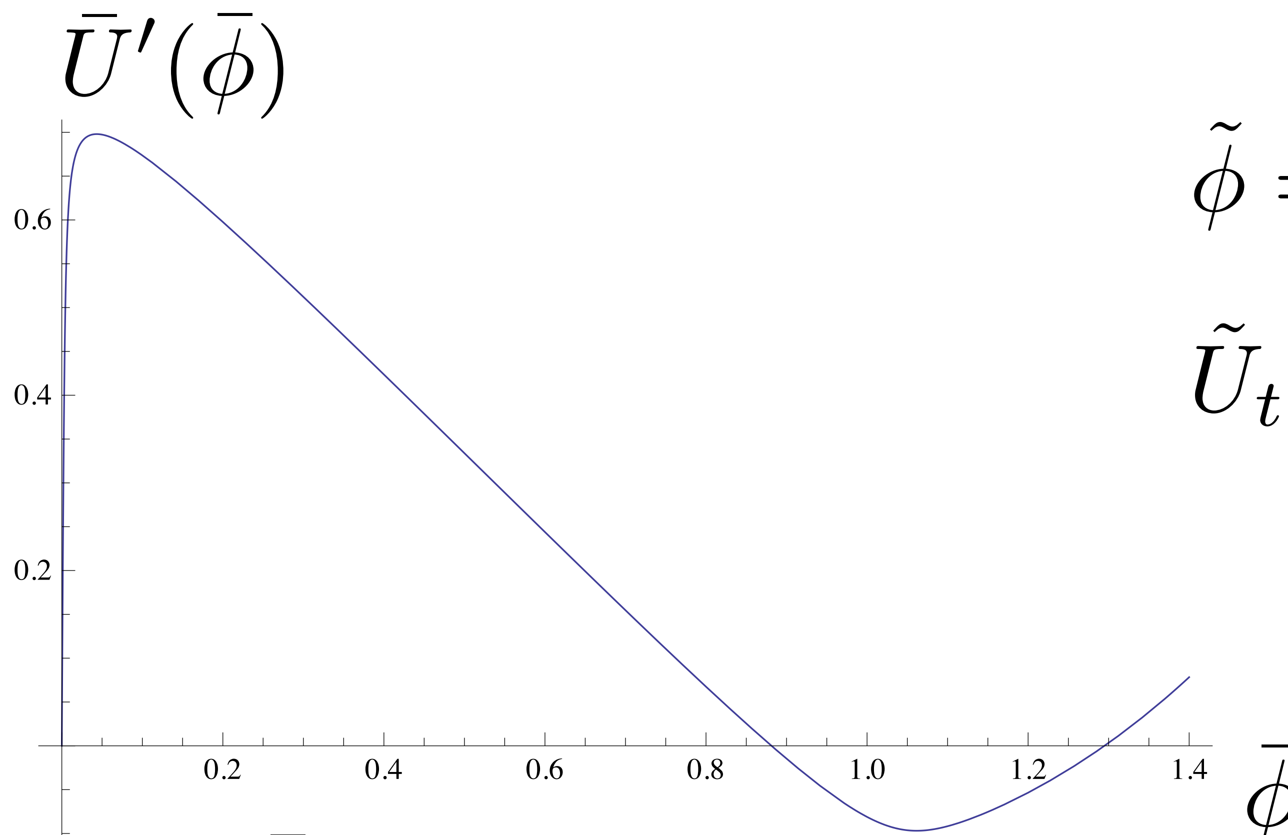


- We try to mimic the behavior of the FP T_2 along the path ABCDEA using a cubic function f(x) depending θ periodically: $f(\theta+2\pi)=f(\theta)$

- Starting from t_2 at (A), we follow this root by continuity all along the path, as indicated with black dots. At $\theta=2\pi$, t_2 has become c_2.

C_2 in the Large-N limit

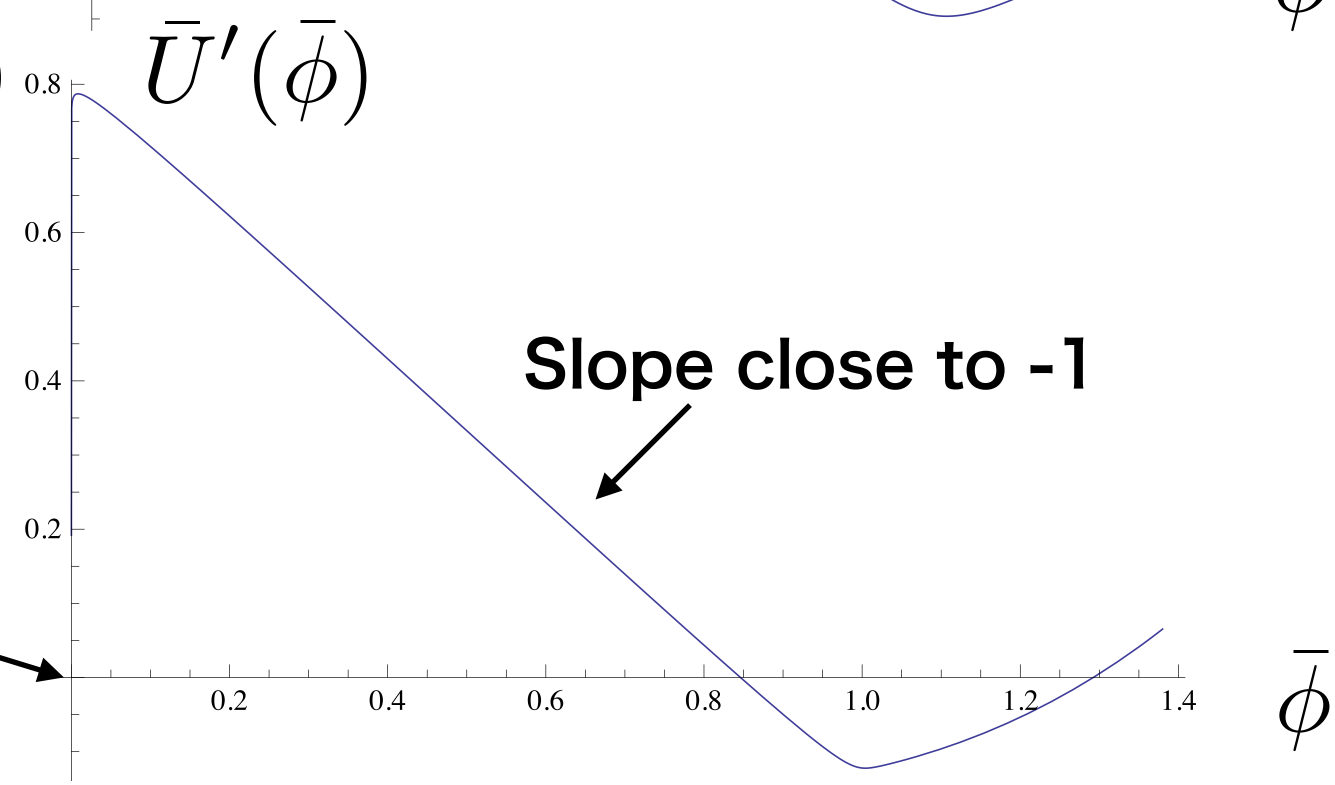
- d=3.2 N=55



$$\tilde{\phi} = \sqrt{N}\bar{\phi}$$

$$\tilde{U}_t = N\bar{U}_t$$

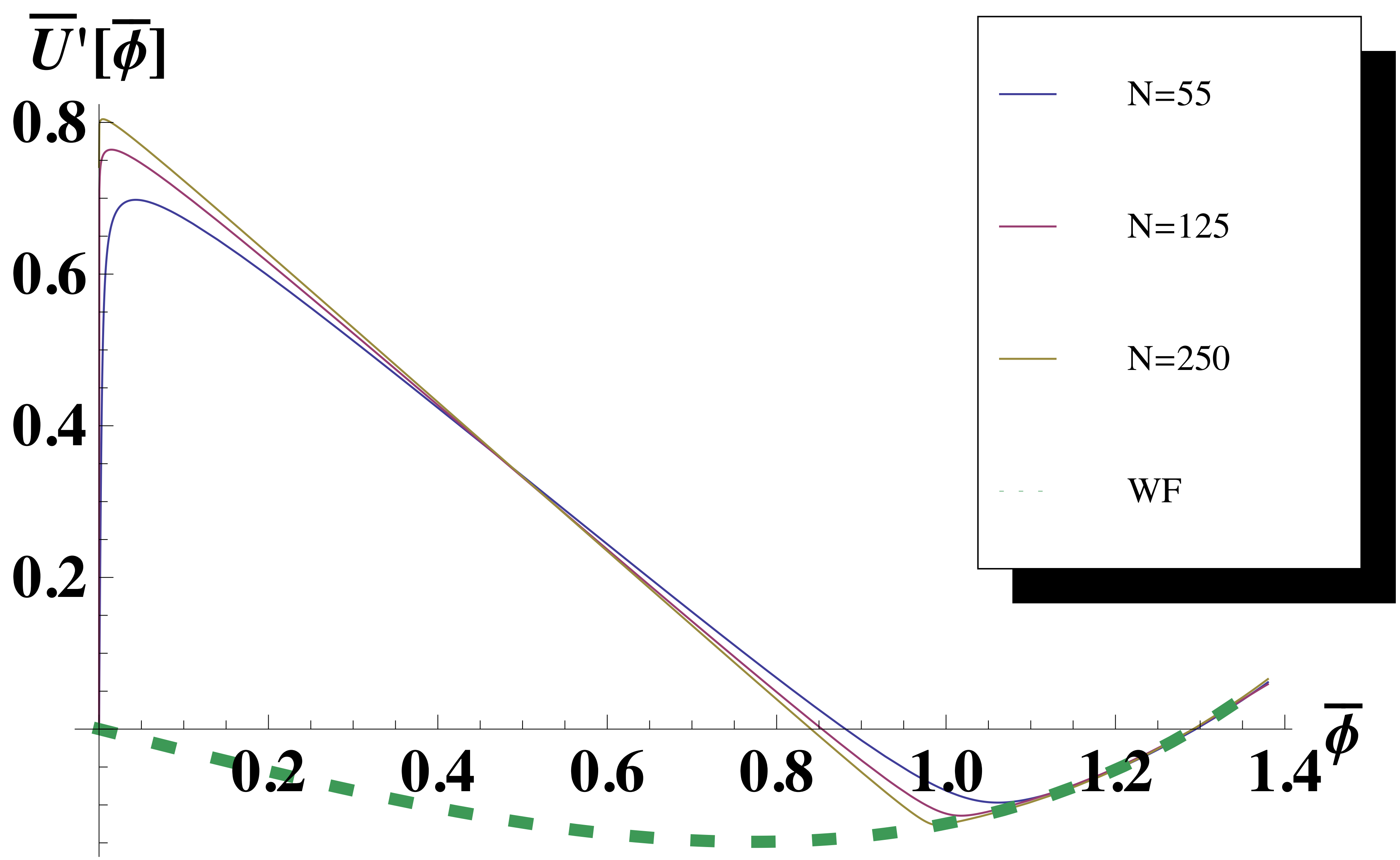
- d=3.2 N=150



From symmetry

$$\bar{U}'(\bar{\phi} = 0) = 0$$

C_2 in the Large-N limit



$O(N) \times O(2)$ model

- Order parameter $\cdots$ $N \times 2$ matrix $\Phi = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$

  $\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = \delta_{ij}$ at the ground state

- Ginzburg-Landau-Wilson Hamiltonian$\cdots$

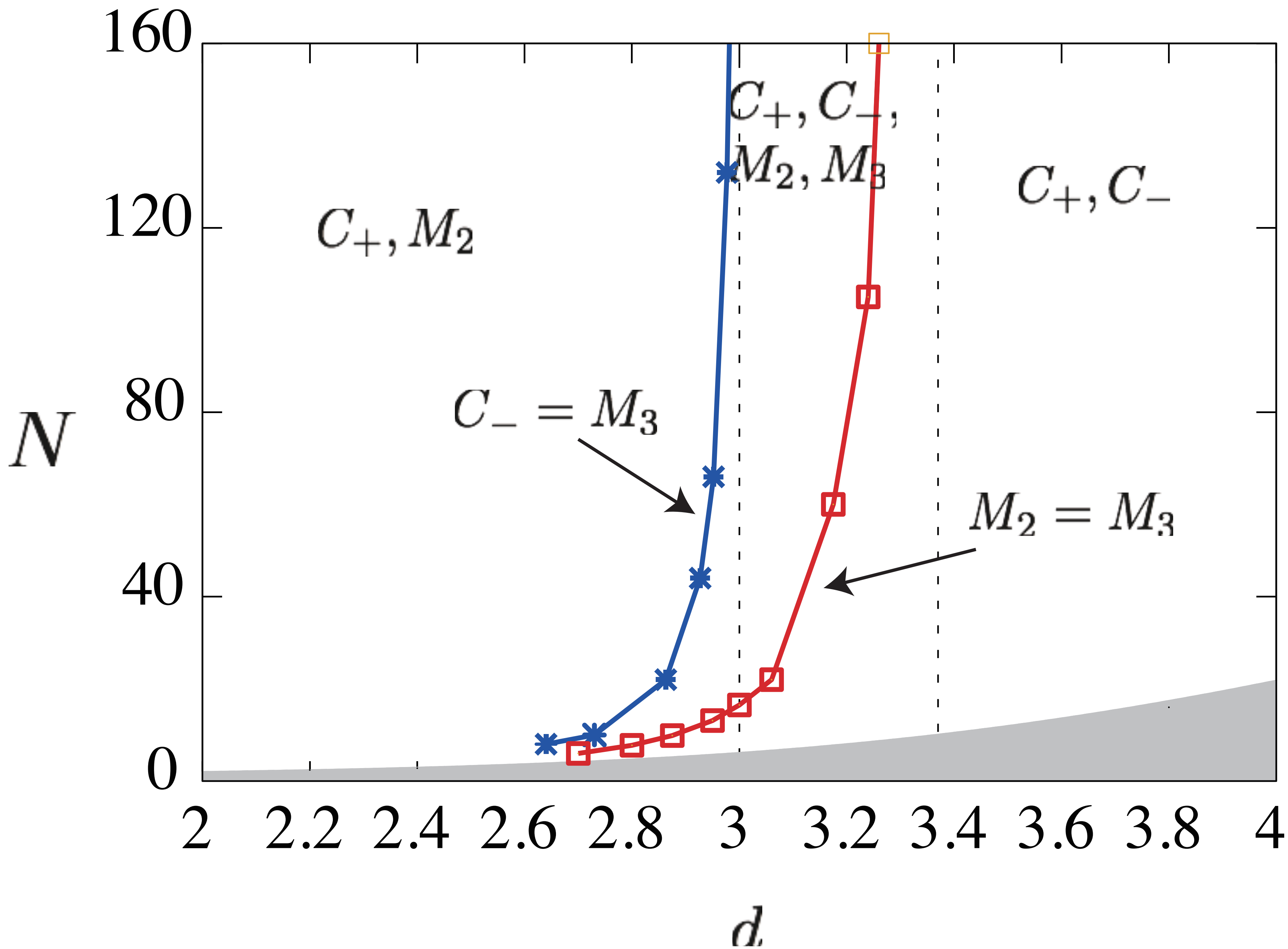
$$H = \int d^d\mathbf{x} \left(\frac{1}{2} \left[(\partial \boldsymbol{\phi}_1)^2 + (\partial \boldsymbol{\phi}_2)^2 \right] + U(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) \right)$$

$U(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ whose minima are given by $\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = const \times \delta_{ij}$

$O(N) \times O(2)$ invariant$\cdots$
$$\begin{aligned} \rho &= \mathrm{Tr}\left({}^t\Phi\Phi\right) \\ \tau &= \tfrac{1}{2}\mathrm{Tr}\left({}^t\Phi\Phi - \rho/2\right)^2 \end{aligned}$$

- Up to the 4-th order $\cdots$ $U(\rho, \tau) = \dfrac{\lambda}{2}(\rho - \kappa)^2 + \mu\tau$

Fixed point structure of $\mathrm{O}(N)\otimes\mathrm{O}(2)$ models



Summary

- We have found that the fixed point structure of both $O(N)$ and $O(N) \times O(2)$ models is much more complicated than widely believed.

- In particular, we have found several nonperturbative FPs in d=3 that were not found previously.

- C_2 and T_2 have double-valued structure in (d,N) space.

- This questions the usual Large-N approach. O(N) models are not soluble in general even at $N = \infty$.